Higher Technological Institute
Computer Science Department

Geometric Transform

Dr Osama Farouk
Dr Ayman Soliman
Dr Adel Khaled

Outline

- Inverse Transformations
- Two-Dimensional Composite Transformations
- Composite Two-Dimensional Translations
- Composite Two-Dimensional Rotations
- Composite Two-Dimensional Scaling
- General Two-Dimensional Pivot-Point Rotation
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Inverse Transformations

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

This produces a translation in the opposite direction

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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$$\mathbf{R}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Negative values for rotation angles generate rotations in a clockwise direction.

$$\mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse matrix generates an opposite scaling transformation.

Two-Dimensional Composite Transformations

- Using matrix representations, we can set up a sequence of transformations as a composite transformation matrix by calculating the product of the individual transformations

- Forming products of transformation matrices is often referred to as a concatenation, or composition of matrices

Two-Dimensional Composite Transformations

- We do pre-multiply the column matrix by the matrices representing any transformation sequence

- since many positions in a scene are typically transformed by the same sequence, it is more efficient to first multiply the transformation matrices to form a single composite matrix

  - Thus, if we want to apply two transformations to point position **P**, the transformed location would be calculated as
  - The coordinate position is transformed using the composite matrix **M**, rather than applying the individual transformations $\mathbf{M}_1$ and then $\mathbf{M}_2$.

$$\begin{aligned}\mathbf{P}' &= \mathbf{M}_2 \cdot \mathbf{M}_1 \cdot \mathbf{P} \\ &= \mathbf{M} \cdot \mathbf{P}\end{aligned}$$

Composite Two-Dimensional Translations

- If two successive translation vectors (t_{1x}, t_{1y}) and (t_{2x}, t_{2y}) are applied to a 2-D coordinate position P, the final transformed location P′ is

$$\mathbf{P}' = \mathbf{T}(t_{2x}, t_{2y}) \cdot \{\mathbf{T}(t_{1x}, t_{1y}) \cdot \mathbf{P}\}$$
$$= \{\mathbf{T}(t_{2x}, t_{2y}) \cdot \mathbf{T}(t_{1x}, t_{1y})\} \cdot \mathbf{P}$$

- The composite transformation matrix for this sequence of translations is

$$\begin{bmatrix} 1 & 0 & t_{2x} \\ 0 & 1 & t_{2y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{1x} \\ 0 & 1 & t_{1y} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{1x} + t_{2x} \\ 0 & 1 & t_{1y} + t_{2y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}(t_{2x}, t_{2y}) \cdot \mathbf{T}(t_{1x}, t_{1y}) = \mathbf{T}(t_{1x} + t_{2x}, t_{1y} + t_{2y})$$

Composite Two-Dimensional Rotations

- Two successive rotations applied to a point P

$$\mathbf{P}' = \mathbf{R}(\theta_2) \cdot \{\mathbf{R}(\theta_1) \cdot \mathbf{P}\}$$
$$= \{\mathbf{R}(\theta_2) \cdot \mathbf{R}(\theta_1)\} \cdot \mathbf{P}$$

- We can verify that two successive rotations are additive:

$$\mathbf{R}(\theta_2) \cdot \mathbf{R}(\theta_1) = \mathbf{R}(\theta_1 + \theta_2)$$

- The composition matrix

$$\mathbf{P}' = \mathbf{R}(\theta_1 + \theta_2) \cdot \mathbf{P}$$

Composite Two-Dimensional Scaling

- For two successive scaling operations in 2-D produces the following composite scaling matrix

$$\begin{bmatrix} s_{2x} & 0 & 0 \\ 0 & s_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{1x} & 0 & 0 \\ 0 & s_{1y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{1x} \cdot s_{2x} & 0 & 0 \\ 0 & s_{1y} \cdot s_{2y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{S}(s_{2x}, s_{2y}) \cdot \mathbf{S}(s_{1x}, s_{1y}) = \mathbf{S}(s_{1x} \cdot s_{2x}, \; s_{1y} \cdot s_{2y})$$

General Two-Dimensional Pivot-Point Rotation

- Graphics package provides only a rotate function with respect to the coordinate origin
- To generate a 2-D rotation about any other pivot point (x_r, y_r), follows the sequence of translate-rotate-translate operations
  1. Translate the object so that the pivot-point position is moved to the coordinate origin
  2. Rotate the object about the coordinate origin
  3. Translate the object so that the pivot point is returned to its original position

General Two-Dimensional Pivot-Point Rotation

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Original Position of Object and Pivot Point

(b) Translation of Object so that Pivot Point (x_r, y_r) is at Origin

(c) Rotation about Origin

(d) Translation of Object so that the Pivot Point is Returned to Position (x_r, y_r)

Problem :

Consider we have a square **O(0, 0), B(4, 0), C(4, 4), D(0, 4)** on which we first apply **T1(scaling transformation) given scaling factor is Sx=Sy=0.5** and then we apply **T2(rotation transformation in clockwise direction)** it by 90*(angle), in last we perform **T3(reflection transformation about origin).**

Ans : The square O, A, C, D looks like :

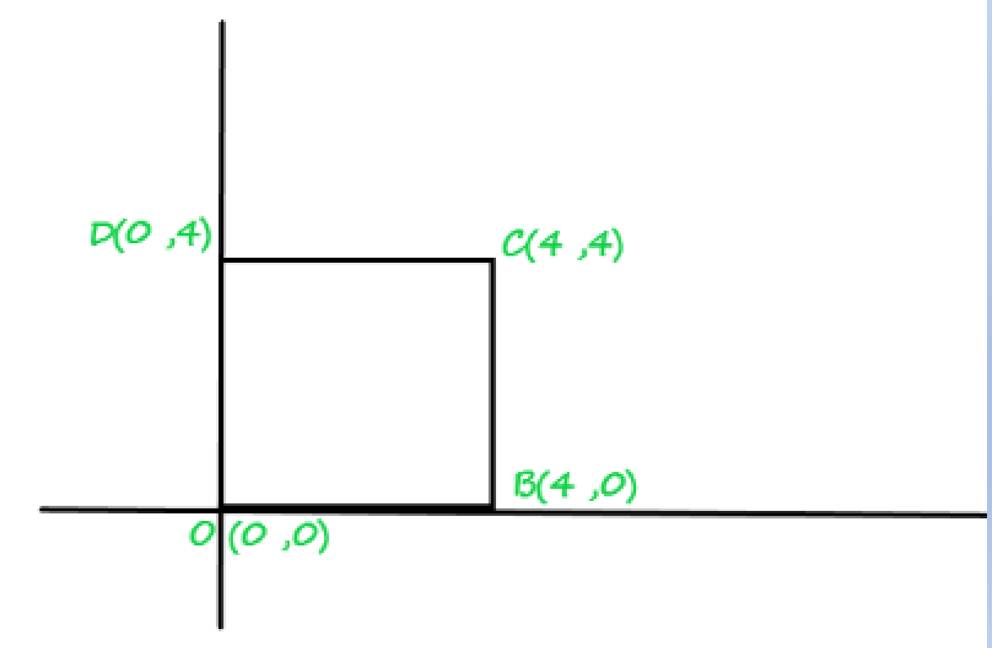

Figure 1

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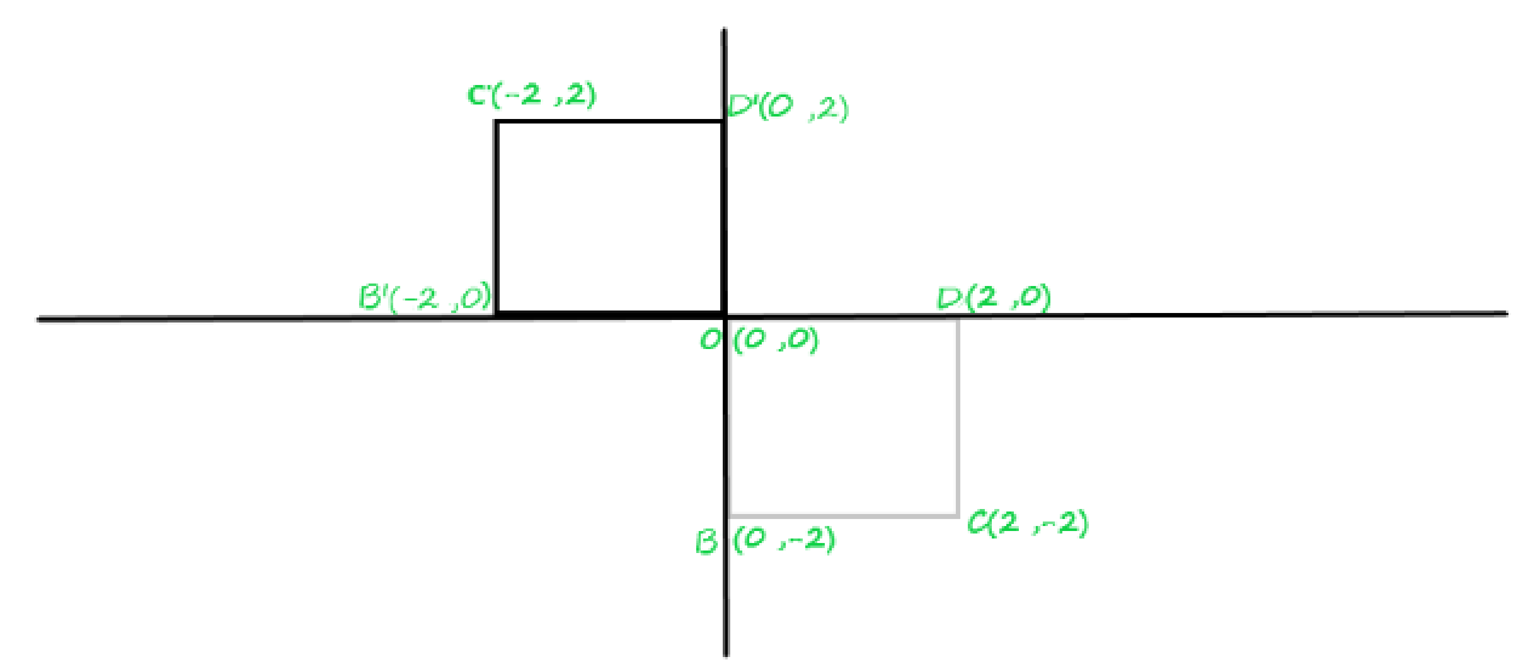


لو هنمشي خطوة خطوة تصغير ثم تدوير

First, we perform **scaling transformation** over a 2-D object :

Representation of scaling condition :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} Sx & 0 \\ 0 & Sx \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

1

For coordinate O(0, 0) :

$$O\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$O\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3

For coordinate C(4, 4) :

$$C\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} * \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$C\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

2

For coordinate B(4, 0)

$$B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} * \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$B\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

4

For coordinate D(0, 4) :

$$D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} * \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$D\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

After scaling

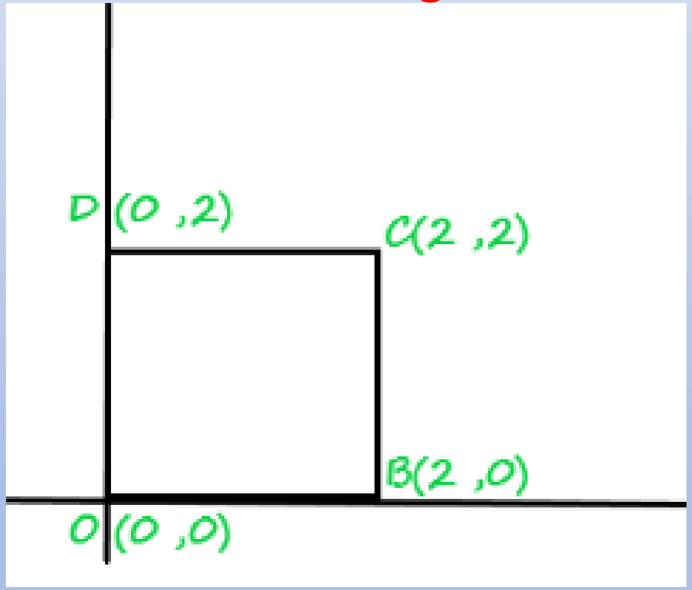


Figure 2

Now, we'll perform rotation transformation in clockwise-direction on Fig.2 by 90^{θ}

The condition of rotation transformation of 2-D object about origin is :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos & sin \\ -sin & cos \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

$\mathrm{Cos}90 = 0$

$\sin 90 = 1$

For coordinate O(0, 0) :

$$O \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$O \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For coordinate B(2, 0) :

$$B \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$B \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

For coordinate C(2, 2) :

$$C \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$C \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

For coordinate D(0, 2) :

$$D \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$D \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

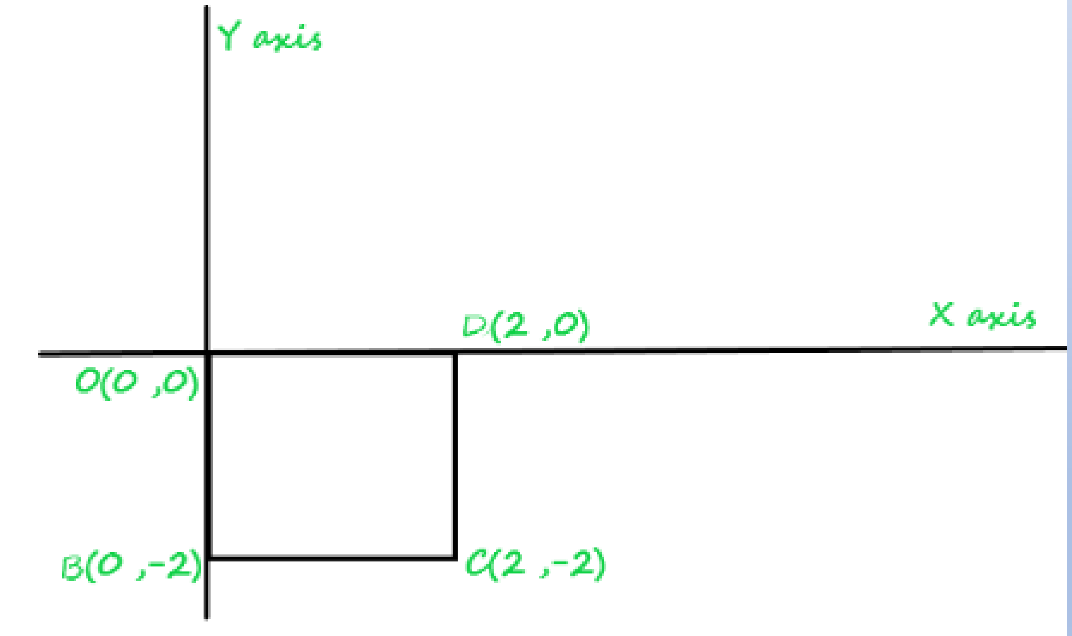


Figure 3

Now, we'll perform third last operation on Fig.3, by reflecting it about origin :

The condition of reflecting an object about origin is (angle is $\pi = 180$)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

For coordinate O(0, 0) :

$$\mathrm{O}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathrm{O}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For coordinate B'(0, 0) :

$$\mathrm{B'}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\mathrm{B'}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

For coordinate C'(0, 0) :

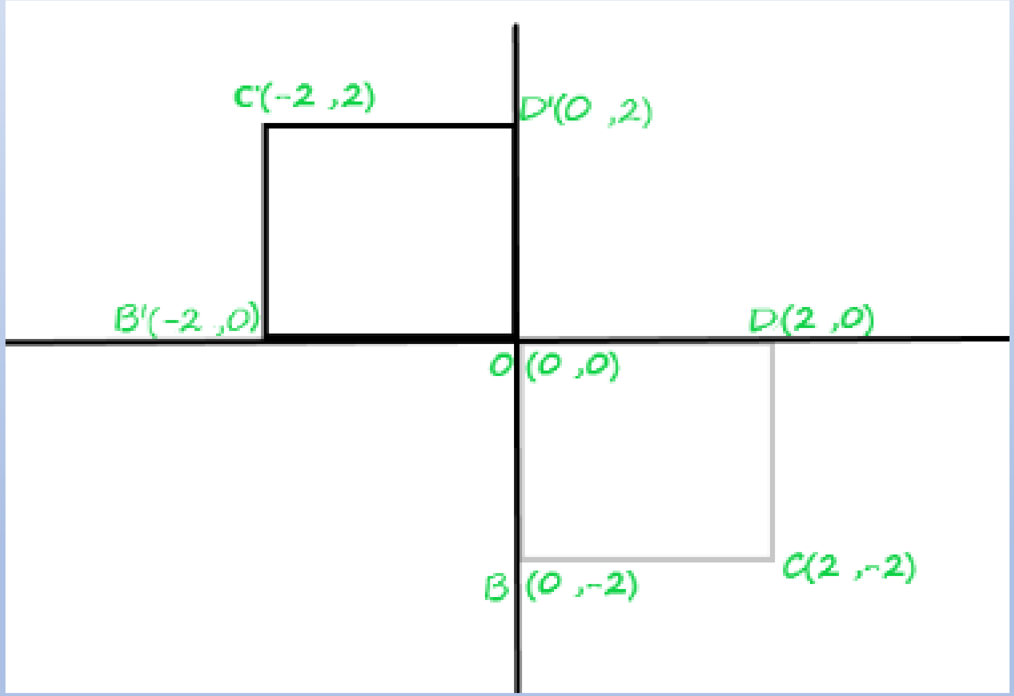
$$\mathrm{C'}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\mathrm{C'}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

For coordinate D'(0, 0) :

$$\mathrm{D'}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\mathrm{D'}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



Final Shape

Solution using Composite transformation : **لو هنعملها بطريقة ال composite matrix مرة واحد يعني**

First we multiplied 2-D matrix conditions of **Scaling transformation** with **Rotation transformation :**

$$[R_1] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$$

Now, we multiplied Resultant 2-D matrix(R_1) with the third last given Reflecting condition of transformation(R_2) to get Resultant(R) :

$$[R] = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix} * \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} R \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$$

Now, we'll applied the Resultant(R) of 2d-matrix at each coordinate of the given object (square) to get the final transformed or modified object.

First transformed coordinate O(0, 0) is :

$$O\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$O\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second, transformed coordinate B'(4, 0) is :

$$B'\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} * \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$B'\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Third transformed coordinate C'(4, 4) is :

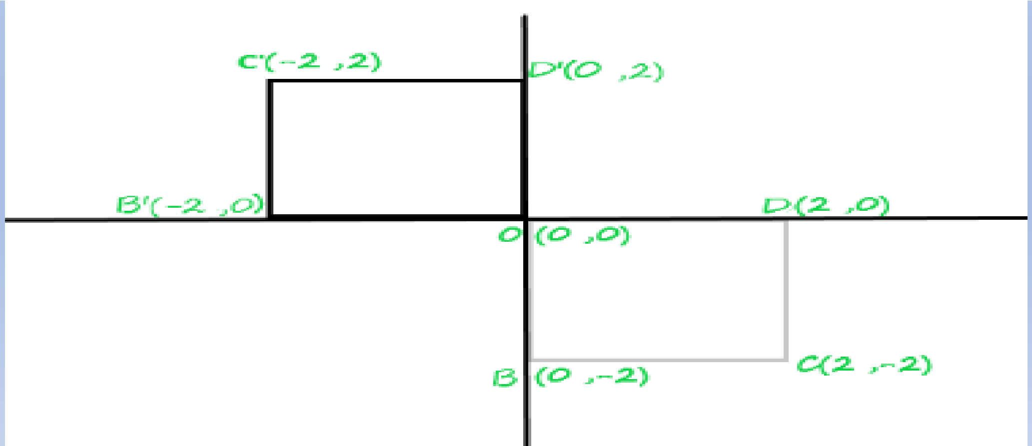
$$C'\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} * \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$C'\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Fourth transformed coordinate D'(0, 4) is :

$$D'\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$D'\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$



End of Lecture
Good Luck!

See you

in next lecture…